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Duo group rings

Howard E. Bell, Yuanlin Li*

Department of Mathematics, Brock University, St. Catharines, Ontario, Canada L2S 3A1

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Abstract

It is shown that the group algebra KG of a torsion group G over a field K is duo if and only if it is reversible.

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1. Introduction and preliminaries

Let R be an associative ring. Call R left (right) duo if every left (right) ideal is an ideal, and call R duo if it is both left and right duo. Define R to be reversible if $\alpha\beta = 0$ implies $\beta\alpha = 0$, and symmetric if $\alpha\beta\gamma = 0$ implies $\alpha\gamma\beta = 0$ for all $\alpha, \beta, \gamma \in R$. Finally, say that R has the "SI" property if $\alpha\beta = 0$ implies $\alpha R\beta = 0$ for all $\alpha, \beta \in R$.

Let k be a commutative ring with identity and G be any group. Using the standard involution $*$ on the group ring kG, defined by $\left(\sum a_i g_i\right)^* = \sum a_i g_i^{-1}$ for all $a_i \in k$ and $g_i \in G$, we see that kG is left duo if and only if it is right duo.

Marks [3] has clarified the relationships among duo, reversible and symmetric rings. Moreover, he proved the following result [3, Proposition 6].

Proposition 1.1. *Let k be a commutative ring with identity, and let G be a finite group. Then the group ring kG is reversible if and only if kG has the "SI" property.*

We note that this result remains valid for an arbitrary group G. We also note that the "SI" property is simply the statement that left annihilators and right annihilators are ideals, and hence it is obvious that duo rings have the "SI" property. It now follows from Proposition 1.1 that if kG is a duo ring, then it is reversible. However, the converse is not true, as the following example shows.

Example 1.2. Let $Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, a^b = a^{-1}\rangle$ be the quaternion group of order 8. The integral group ring $\mathbb{Z}Q_8$ is a reversible ring, but not a duo ring.

* Corresponding author.
E-mail addresses: hbell@brocku.ca (H.E. Bell), yli@brocku.ca (Y. Li).

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Proof. It follows from Theorem 3.1 in [1] that the rational group algebra $\mathbb{Q}Q_8$ is reversible. As a subring of the rational algebra, clearly, the integral group ring $\mathbb{Z}Q_8$ is reversible.

Next we prove that $R = \mathbb{Z}Q_8$ is not a duo ring by showing that the left ideal $R(a + 2b)$ generated by $a + 2b$ is not a right ideal. To see this, we shall show that $(a + 2b)a$ is not in $R(a + 2b)$. Suppose that $(a + 2b)a = \beta(a + 2b)$, where $\beta = \sum_{i=0}^{3} a_i a^i + \left(\sum_{j=4}^{7} a_j a^{j-4}\right) b \in \mathbb{Z}Q_8$. Then we have $a^2 + 2a^3 b = (a_3 + 2a_6) + (a_0 + 2a_7)a + (a_1 + 2a_4)a^2 + (a_2 + 2a_5)a^3 + (2a_0 + a_5)b + (2a_1 + a_6)ab + (2a_2 + a_7)a^2 b + (2a_3 + a_4)a^3 b$.

Comparing the coefficients of $1, a^2, ab$ and a^3b on both sides of the above equation, we obtain the following system.

$$\begin{aligned} a_3 + 2a_6 &= 0 \\ a_1 + 2a_4 &= 1 \\ 2a_1 + a_6 &= 0 \\ 2a_3 + a_4 &= 2. \end{aligned} \tag{1.1}$$

Eliminating a_1 and a_3, we get $a_4 - 4a_6 = 2$ and $4a_4 - a_6 = 2$. Thus $a_6 = -\frac{6}{15} \notin \mathbb{Z}$. Therefore, $(a + 2b)a \notin \mathbb{Z}Q_8$ and thus we conclude that $\mathbb{Z}Q_8$ is not a duo ring. $\square$

We remark that $\mathbb{Z}Q_8$ is, in fact, a symmetric ring. Hence this example shows that “symmetric” does not imply “duo”.

As mentioned earlier, if the group ring kG is duo, then it is reversible. All reversible group rings of torsion groups over fields are characterized by Gutan and Kisielewicz [1]. In a recent paper [2], the second author and Parmenter investigated the reversible group rings kG of torsion groups G over commutative rings.

In this paper, we shall study when the group algebras KG of torsion groups over fields are duo rings. Our main result is that the group algebra KG is a duo ring if and only if KG is reversible. It was shown in [1] that if the group algebra KG of a torsion group G is reversible, then either G is an Abelian group, or G is a Hamiltonian group and the characteristic of K is 0 or 2. If G is Abelian, then clearly RG is both reversible and duo. Thus we may always assume that K has characteristic 0 or 2 and G is a Hamiltonian group, i.e. $G = Q_8 \times E_2 \times E_2'$, where Q_8 is the quaternion group of order 8, E_2 is an elementary Abelian 2-group, and E_2' is an Abelian group all of whose elements are of odd order. We first deal with the special case when $G = Q_8$, in the next two sections. The general case when G is a Hamiltonian group will be handled in the last section. As mentioned earlier, $Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1} \rangle$. Our other notation follows that in [4].

2. Characteristic 0

Let $G = Q_8$ and K be a field of characteristic 0. The following result can be derived from some known results in group rings along with the Wedderburn decomposition of KQ_8.

Theorem 2.1. *The following statements are equivalent:*

(1) $R = KQ_8$ *is duo.*
(2) $R = KQ_8$ *is reversible.*
(3) *The equation* $1 + x^2 + y^2 = 0$ *has no solutions in* K.

Proof. It follows from Lemma 7.4.9 in [4] that KQ_8 has the Wedderburn decomposition $KQ_8 \cong K \oplus K \oplus K \oplus K \oplus H(K)$, where $H(K)$ is the quaternion algebra over K. By Theorem 7.4.6 in [4], $H(K)$ is either a division ring D or the 2×2 full matrix ring $M_2(K)$ over K, and the latter possibility arises if and only if $x^2 + y^2 + 1 = 0$ has solutions in K. Thus KQ_8 is duo (or reversible) iff $H(K)$ is duo (or reversible) iff $H(K)$ is a division ring iff $x^2 + y^2 + 1 = 0$ has no solutions in K. This finishes the proof. $\square$

Remark 2.2. (2) $\Leftrightarrow$ (3) is the result of Theorem 3.1 in [1].

3. Characteristic 2

In this section, we characterize when KQ_8 is duo, where K is a field with characteristic 2. Necessary and sufficient conditions for KQ_8 to be duo are given in the following theorem.

Theorem 3.1. *The following statements are equivalent:*

(1) KQ_8 *is duo.*
(2) KQ_8 *is reversible.*
(3) *The equation* $1 + x + x^2 = 0$ *has no solutions in* K.

Note that (1) ⇒ (2) is clear, and (2) ⇒ (3) follows from Theorem 4.1 in [1]. We shall prove (3) ⇒ (1) by using case-by-case analysis.

For any $v \in R = KQ_8$, we shall prove that Rv is an ideal, and so KQ_8 is left duo. As mentioned in the introduction, KQ_8 is therefore duo. Write $v = \sum_{i=0}^{3} a_i a^i + \left(\sum_{j=4}^{7} a_j a^{j-4}\right) b$, where each $a_i, a_j \in K$. If the augmentation $\epsilon(v)$ of v is not zero, then v is a unit, and so clearly Rv is an ideal. Next we assume that $\epsilon(v) = 0$, and we shall verify that both va and vb are in Rv. The crucial step is to show that $va + av = \beta v$ for some $\beta = \sum_{i=0}^{3} a'_i a^i + \left(\sum_{j=4}^{7} a'_j a^{j-4}\right) b \in KG$. Simplifying and then comparing the coefficients of group elements on both sides of the above equation, we obtain the following system.

$$\begin{aligned}
&a_0a'_0 + a_3a'_1 + a_2a'_2 + a_1a'_3 + a_6a'_4 + a_7a'_5 + a_4a'_6 + a_5a'_7 = 0\\
&a_1a'_0 + a_0a'_1 + a_3a'_2 + a_2a'_3 + a_5a'_4 + a_6a'_5 + a_7a'_6 + a_4a'_7 = 0\\
&a_2a'_0 + a_1a'_1 + a_0a'_2 + a_3a'_3 + a_4a'_4 + a_5a'_5 + a_6a'_6 + a_7a'_7 = 0\\
&a_3a'_0 + a_2a'_1 + a_1a'_2 + a_0a'_3 + a_7a'_4 + a_4a'_5 + a_5a'_6 + a_6a'_7 = 0\\
&a_4a'_0 + a_7a'_1 + a_6a'_2 + a_5a'_3 + a_0a'_4 + a_1a'_5 + a_2a'_6 + a_3a'_7 = a_5 + a_7\\
&a_5a'_0 + a_4a'_1 + a_7a'_2 + a_6a'_3 + a_3a'_4 + a_0a'_5 + a_1a'_6 + a_2a'_7 = a_4 + a_6\\
&a_6a'_0 + a_5a'_1 + a_4a'_2 + a_7a'_3 + a_2a'_4 + a_3a'_5 + a_0a'_6 + a_1a'_7 = a_5 + a_7\\
&a_7a'_0 + a_6a'_1 + a_5a'_2 + a_4a'_3 + a_1a'_4 + a_2a'_5 + a_3a'_6 + a_0a'_7 = a_4 + a_6.
\end{aligned} \tag{3.1}$$

The augmented matrix of the above system is as follows:

$$\begin{bmatrix} A & B & O \\ C & D & Y \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} a_0 & a_3 & a_2 & a_1 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & a_1 & a_0 & a_3 \\ a_3 & a_2 & a_1 & a_0 \end{bmatrix}, B = \begin{bmatrix} a_6 & a_7 & a_4 & a_5 \\ a_5 & a_6 & a_7 & a_4 \\ a_4 & a_5 & a_6 & a_7 \\ a_7 & a_4 & a_5 & a_6 \end{bmatrix}, C = \begin{bmatrix} a_4 & a_7 & a_6 & a_5 \\ a_5 & a_4 & a_7 & a_6 \\ a_6 & a_5 & a_4 & a_7 \\ a_7 & a_6 & a_5 & a_4 \end{bmatrix},$$

$$D = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_3 & a_0 & a_1 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_1 & a_2 & a_3 & a_0 \end{bmatrix}, O \text{ is the } 4 \times 1 \text{ zero matrix and } Y = \begin{bmatrix} a_5 + a_7 \\ a_4 + a_6 \\ a_5 + a_7 \\ a_4 + a_6 \end{bmatrix}.$$

From now on, we always assume that $1 + x + x^2 = 0$ has no solutions in K.

Lemma 3.2. *Let* $v = \sum_{i=0}^{3} a_i a^i + \left(\sum_{j=4}^{7} a_j a^{j-4}\right) b \in KQ_8$ *with* $\epsilon(v) = 0$. *If* $\sum_{i=0}^{3} a_i \neq 0$, *then* $va, vb \in Rv$ *and thus* Rv *is an ideal.*

Proof. Since $\sum_{i=0}^{3} a_i \neq 0$, $u = \sum_{i=0}^{3} a_i a^i$ is a unit. Thus $v = u\left(1 + u^{-1}\left(\sum_{j=4}^{7} a_j a^{j-4}\right) b\right)$. If $au^{-1}v + u^{-1}va \in Ru^{-1}v \subseteq Rv$, then $u^{-1}va \in Rv$ and so $va \in Rv$. By replacing v by $u^{-1}v$, we may assume that $v = 1 + \left(\sum_{j=4}^{7} a_j a^{j-4}\right) b$.

We first show that $va \in Rv$. Note that the augmented matrix of system (3.1) is now reduced to

$$\begin{bmatrix} I & B & O \\ C & I & Y \end{bmatrix}.$$

By performing the obvious row operations to reduce C to O, we see that this augmented matrix is row-equivalent to

$$\begin{bmatrix} I & B & O \\ O & D' & Y' \end{bmatrix}, \quad \text{where } D' = \begin{bmatrix} 1 & 1+t & 1 & 1+t \\ 0 & t^2 & 0 & t^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ with } 1+w = a_4+a_6 = 1+a_5+a_7 \text{ and}$$

$$t = 1+w+w^2, \text{ and } Y' = \begin{bmatrix} w \\ 1+wt \\ 0 \\ 0 \end{bmatrix}.$$

This shows that the system (3.1) has a solution if and only if $t \neq 0$. By the assumption, $t = 1+w+w^2 \neq 0$, and thus $va \in Rv$. Next we show that $vb \in Rv$. Noting that v can be rewritten as $v = 1+a_4b+a_6b^3+(a_7b+a_5b^3)a$ and that a, b are symmetric in the generating set of Q_8, we see that if $1+a_4+a_6 \neq 0$, then $vb \in Rv$. Thus we may assume that $1+a_4+a_6 = 0$. Then $1+a_5+a_7 = a_4+a_6 \neq 0$. Note that $v = 1+a_5(ab)+a_7(ab)^3+(a_4+a_6(ab))b$, so a similar argument shows that $vab \in Rv$. Therefore, $vb = va^3(ab) \in Rvab \subseteq Rv$ and thus Rv is an ideal. □

Remark 3.3. We note that since $v = (a_0+a_4b+a_2b^2+a_6b^3)+(a_1+a_7b+a_3b^2+a_5b^3)a$, if $a_0+a_2+a_4+a_6 \neq 0$, by the symmetry of a and b, Lemma 3.2 shows that Rv is an ideal. Next we may assume that both $a_0+a_1+a_2+a_3$ and $a_0+a_2+a_4+a_6$ are zero. Since $\epsilon(v) = 0$, the above is equivalent to $a_0+a_2 = a_4+a_6 = a_1+a_3 = a_5+a_7$.

Lemma 3.4. *Let $v = \sum_{i=0}^{3} a_i a^i + \left(\sum_{j=4}^{7} a_j a^{j-4}\right) b \in KQ_8$ such that $a_0+a_2 = a_4+a_6 = a_1+a_3 = a_5+a_7 = s\,(*)$. Then $va, vb \in Rv$ and thus Rv is an ideal.*

Proof. If $s = 0$, then $v = \alpha(1+a^2)$ where $\alpha \in KQ_8$. It can then be easily shown that v is a central element and hence Rv is an ideal.

Next assume that $s \neq 0$. We first show that both the coefficient matrix and the augmented matrix of system (3.1) have the same rank 4 and so $va \in Rv$.

We shall apply row and column reductions on the augmented matrix.

Adding row i to row $(i+2)$ for $i = 1, 2, 5, 6$ in the augmented matrix, and then simplifying resulting matrix by using row reductions and conditions $(*)$, we reduce the augmented matrix to

$$\begin{bmatrix} A_1 \\ O_{3\times 9} \end{bmatrix}, \quad \text{where } O \text{ is a zero matrix and } A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ a_0 & a_3 & a_2 & a_1 & a_6 & a_7 & a_4 & a_5 & 0 \\ a_1 & a_0 & a_3 & a_2 & a_5 & a_6 & a_7 & a_4 & 0 \\ a_4 & a_7 & a_6 & a_5 & a_0 & a_1 & a_2 & a_3 & s \\ a_5 & a_4 & a_7 & a_6 & a_3 & a_0 & a_1 & a_2 & s \end{bmatrix}.$$

Applying the similar column operations (symmetric with the above row operations) and simplifying the resulting matrix, we obtain

$$\begin{bmatrix} A_2 & O_{5\times 3} \\ O_{3\times 6} & O_{3\times 3} \end{bmatrix}, \quad \text{where } A_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & a_0 & a_3 & a_6 & a_7 & 0 \\ 1 & a_1 & a_0 & a_5 & a_6 & 0 \\ 1 & a_4 & a_7 & a_0 & a_1 & 1 \\ 1 & a_5 & a_4 & a_3 & a_0 & 1 \end{bmatrix}.$$

Clearly we need only work on A_2. Next adding rows 2, 3, 4 to row 5 of A_2 and then adding columns 2, 3, 4 to column 5 of the resulting matrix, we get

$$A_3 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & a_0 & a_3 & a_6 & t+s & 0 \\ 1 & a_1 & a_0 & a_5 & t+s & 0 \\ 1 & a_4 & a_7 & a_0 & t+s & 1 \\ 0 & t & t & t & 0 & 0 \end{bmatrix},$$

where $t = a_0+a_1+a_4+a_5 = a_0+a_3+a_4+a_7 = a_0+a_3+a_5+a_6 = a_0+a_1+a_6+a_7$.

Deleting the last row and the fifth column of A_3 and then using row reduction, we finally obtain

$$A_4 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & a_3 + a_0 & a_6 + a_0 & 0 \\ 0 & 0 & s & t + s & 0 \\ 0 & 0 & t & s & 1 \end{bmatrix}.$$

Note that the determinant $\begin{vmatrix} s & t+s \\ t & s \end{vmatrix} = s^2((\frac{t}{s})^2 + \frac{t}{s} + 1) \neq 0$ because $x^2 + x + 1 = 0$ has no solutions in K. Thus both matrix A_4 and the matrix consisting of the first four columns of A_4 have the same rank 4. Consequently, both the coefficient matrix and the augmented matrix of system (3.1) have the same rank 4 as claimed, so $va \in Rv$. Next we note that conditions $(*)$ are symmetric about a and b. By symmetry, we conclude that $vb \in Rv$. Therefore, Rv is an ideal. □

We note that, in fact, we just showed that KQ_8 is duo if and only if $x^2 + x + 1 = 0$ has no solutions in K.

Remark 3.5. The referee has pointed out that the result (2) $\Rightarrow$ (3) in Theorem 3.1 was observed (but not published) by Lam in 2001.

4. General case

In this section, we deal with the general case when KG is a group algebra over a torsion group. We first state the following straightforward result.

Lemma 4.1. *Let $R = R_1 \times R_2$. Then R is (left or right) duo if and only if both R_1 and R_2 are (left or right) duo.*

Here is the main result:

Theorem 4.2. *Let K be a field and let G be a torsion group. Then KG is a duo ring if and only if one of the following conditions holds.*

(1) *G is Abelian.*
(2) *$G = Q_8 \times E_2 \times E_2'$ is Hamiltonian, the characteristic of K is 0 and the equation $1 + x^2 + y^2 = 0$ has no solutions in any cyclotomic field $K(\xi_d)$ for any odd d which is an order of an element of E_2'.*
(3) *$G = Q_8 \times E_2'$, the characteristic of K is 2 and the equation $1 + x + x^2 = 0$ has no solutions in any cyclotomic field $K(\xi_d)$ for any odd d which is an order of an element of E_2'.*

Proof. We need only consider that G is a non-Abelian torsion group. If KG is duo, then it is reversible. Thus $G = Q_8 \times E_2 \times E_2'$ and char(K) is 0 or 2. We first assume that G is finite. If char(K) is 0, then it follows from a theorem of Perlis and Walker [5, Prop. II. 2.6] that

$$KE_2' \cong \left(\prod_{d||E_2'|} K(\xi_d)^{a_d} \right)$$

where ξ_d is a primitive dth root of unity over K and a_d is equal to the number of elements of order d in E_2' divided by the degree of $K(\xi_d)$ over K, i.e. $a_d = n_d/[K(\xi_d) : K]$. Clearly $K(\xi_d)C_2 \cong K(\xi_d) \times K(\xi_d)$.

We now have

$$K(E_2 \times E_2') = (KE_2')(E_2) \cong \prod_{d||E_2'|} K(\xi_d)^{a_d}(E_2) \cong \prod \prod_{d||E_2'|} K(\xi_d)^{a_d},$$

and thus

$$KG \cong \prod \prod_{d||E_2'|} K(\xi_d)^{a_d}(Q_8) \cong \prod \prod_{d||E_2'|} (K(\xi_d)^{a_d} Q_8).$$

It follows from Lemma 4.1 that KG is duo iff each $(K(\xi_d)Q_8)$ is duo iff $1 + x^2 + y^2 = 0$ has no solutions in any cyclotomic field $K(\xi_d)$ for any odd d which is an order of an element of E_2' by Theorem 2.1.

If char(K) is 2, then the assumption that $KG = K(Q_8 \times E_2 \times E_2')$ is duo, and hence reversible, implies that $G = Q_8 \times E_2'$ [1, Theorem 5.1(ii)]. Thus, by Theorem 3.1 $KG \cong \prod_{d||E_2'|} K(\xi_d)^{a_d}(Q_8)$ is duo iff $1 + x + x^2 = 0$ has no solutions in any cyclotomic field $K(\xi_d)$ for any odd d which is an order of an element of E_2'.

We now consider that $G = Q_8 \times E_2 \times E_2'$ is a torsion group, not necessarily finite. We claim that KG is duo iff KH is duo for every finite subgroup H of G.

If KG is duo, then KG is reversible and therefore KH is reversible for every finite subgroup H of G. By what we just proved and [1, Theorem 5.1], we conclude that KH is duo. Conversely, assume that KH is duo for every finite subgroup H of G. For any $v \in KG$ and $g \in G$, the subgroup H generated by the support of v and g is finite, so $vg \in (KH)v \subseteq (KG)v$. This shows that $(KG)v$ is an ideal for any $v \in KG$, and thus KG is duo as desired. □

As a consequence, we have the following:

Corollary 4.3. *Let K be a field and let G be a torsion group. Then KG is duo if and only if KG is reversible.*

It is easy to see that the equation $x^2 + x + 1 = 0$ has a solution in a field K of characteristic 2 if and only if K has a subfield isomorphic to $GF(2^2)$. Thus we have the following corollary.

Corollary 4.4. *Let K be a field with char $K = 2$, and let G be a non-Abelian torsion group. Then KG is duo if and only if $G = Q_8 \times E_2'$ and each cyclotomic field $K(\xi_d)$ in the Wedderburn decomposition of KE_2' has no subfields isomorphic to $GF(2^2)$.*

We remark that if a field K of char $= 2$ has no subfields isomorphic to $GF(2^2)$, then every finite subfield of K must be isomorphic to $GF(2^d)$ for some odd integer d.

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